

"CONSUMER THEORY WITH NONCONVEX CONSUMPTION
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Consumer Theory with NonConvex Consumption
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Abstract:

In this paper we consider a very general formulation for the consumption set.

The traditional formulation and the allowance for a consumption technology may be taken as special cases. We allow for nonconvexities in the consumption set which arise, for example, due to the existence of goods that may be consumed only in discrete units (indivisible commodities).

It seems to us that there is not in the literature an agreement of what will be the influence of these refinements for the main conclusions of the consumer theory.

We arrive at results similar to the conventional consumer's theory. Demands add up, are homogeneous of degree zero in prices and income and compensated demands and prices are negatively correlated.

We also obtain analogues of Shephard's Lemma, Roy's Identity and Slutsky Equation. These results provide a basis for integrability and welfare analysis.

0 - Introduction

In its traditional form, Consumer Theory restricts Marshallian demands ($x = x(p; I)$ where p denotes the vector of prices and I denotes the exogenous income) to satisfy the following properties assuming differentiability (to assure the differentiability of demands one needs some technical assumptions: $u \in C^2$, is monotonous, strictly quase-concave and satisfies some boundary conditions):

P1: (Cournot Aggregation)

$$x_i + \sum_{j=1}^n p_j \frac{\partial x_j}{\partial p_i} = 0 \quad \text{for all } i$$

P2: (Engel Aggregation)

$$\sum_{i=1}^n p_i \frac{\partial x_i}{\partial I} = 1$$

P1 and P2 are usually referred as "adding up" conditions since they follow from the budget constraint and no satiation.

P3: (Homogeneity)

$$\sum_{i=1}^n p_i \frac{\partial x_j}{\partial p_i} + I \frac{\partial x_j}{\partial I} = 0 \quad \text{for all } j$$

From Marshallian demand functions one may build the Slutsky matrix. Let:

$$s_{ij} = \frac{\partial x_i}{\partial p_j} + x_j \frac{\partial x_i}{\partial I} \quad \text{for all } i \text{ and } j \quad (\text{Slutsky equation})$$

and define the Slutsky matrix as:

$$S = \begin{bmatrix} s_{11} & s_{12} & \dots & s_{1n} \\ \vdots & \vdots & & \vdots \\ s_{n1} & s_{n2} & \dots & s_{nn} \end{bmatrix}$$

One may now state the last two properties:

P4: (Symmetry)

$$s_{ij} = s_{ji} \quad \text{for all } i \text{ and } j$$

P5: (Negativity)

S is a negative semi-definite matrix:

$$v^T S v \leq 0 \quad \text{for all } v$$

and

$$p^T S = 0$$

We know that any system of functions that satisfies P1-P5 is a system of Marshallian demand functions which can be obtained by maximizing some utility function u^* (Deaton and Muellbauer, 1980, p.50). This solution to the integrability problem assures that P1-P5 exhaust the empirical implications of utility maximization under the regularity assumptions above.

The question we address is: How do P1-P5 generalize when, for instance, discrete choice is allowed?

It seems to us that there is not in the literature an agreement of what will be the influences of these refinements for the main conclusions of Consumer's Theory.

In Section 1 we present the basic notation and assumptions. In Section 2 we explore the implications of continuity and non satiation of preferences in at least one product. We derive the adding and homogeneity properties and some duality results. In Section 3 we explore the implications of rational behavior and prove the negativity property and derive analogues to Shephard's Lemma, Slutsky Equation and Roy's Identity. We briefly address the integrability problem.

In the final version we plan to include Section 4 in which we show how welfare analysis may be performed in this context. We will also include a conclusion with a summary of the findings of the paper and suggestions for further work, in particular, we plan to assess the possibility of performing empirical research in this area.

1 - Basic Notation and Assumptions

Problem (U): Utility Maximization

$$\begin{aligned} v(p, I) &= \max_x u(x) && (v(p, I) - \text{indirect utility function}) \\ \text{s.t.} & \\ px &\leq I \\ x &\in D \subseteq \mathbb{R}_+^n \end{aligned}$$

where $u(\cdot)$ is continuous, p is a vector of prices ($p \gg 0$), I is exogenous income ($I > 0$) and $D \cap \{x: px \leq I\}$ is a compact set. To simplify some proofs we will assume that D is a compact set.

We assume nonsatiation and perfect divisibility in at least one good (in the remainder of this paper we will take to be the first one, without loss of generality) to assure that the budget constraint is binding and that some duality results hold.

The set of optimal solutions to (U) - Marshallian demands - will be denoted as $\{x(p, I)\}$. Note that $\{x(p, I)\} \neq \emptyset$ for all (p, I) considered (by Weierstrass theorem see Sydsæter, 1981, p.219).

Problem (E): Expenditure Minimization

$$\begin{aligned} e(p, \bar{u}) &= \min_x px && (e(p, \bar{u}) - \text{expenditure function}) \\ \text{s.t.} & \\ u(x) &\geq \bar{u} \\ x &\in D \end{aligned}$$

The set of optimal solutions to (E) is denoted as $\{h(p, \bar{u})\}$ - Hicksian or compensated demands. Note that $\{h(p, \bar{u})\} \neq \emptyset$ under the conditions above if $\bar{u} \in \{\text{range of } u(\cdot)\}$.

The textbooks formulations (e.g. Varian, 1984, chap.2, Deaton and Muellbauer, 1980, chap.2, Phelps, 1983, chap.1) are particular cases of the formulations above with $D = \mathbb{R}_+^n$.

(implicitly). Debreu (1959) assumes D to be a convex set. Convexity of D implies that all commodities are continuously divisible.

With the formulation above we can handle, for instance, the case of the existence of goods that can only be consumed in discrete units as $D = \{ (x_1, x_2, \dots, x_n) : (x_2, \dots, x_n) \in \mathbb{Z}^{n-1} \text{ and } x_1 \in \mathbb{R} \} \cap \{ x : px \leq I \}$ is a compact set (\mathbb{Z} is the set of integers).

The consumption set D may also reflect, for example, restrictions derived from the existence of a consumption technology which may be nonconvex.

2 - Implications of continuity and nonsatiation in at least one good.

As we assume above $x_1 \in \mathbb{R}$ (x_1 is the quantity of good 1) and there is nonsatiation in good 1. With these two assumptions we can prove the following:

PROPOSITION 2.1.:

(a) [adding up]: $px = I \quad x \in \{x(p, I)\}$.

(b) $\{x(p, e(p, \bar{u}))\} = \{h(p, \bar{u})\}$

(c) $v(p, e(p, \bar{u})) = \bar{u}$

if $\lambda > 0$ one has

(d) [Homogeneity (1)]: $v(\lambda p, \lambda I) = v(p, I)$

(e) [Homogeneity (2)]: $e(\lambda p, \bar{u}) = \lambda e(p, \bar{u})$

(f) [Homogeneity (3)]: $\{x(\lambda p, \lambda I)\} = \{x(p, I)\}$

(g) [Homogeneity (4)]: $\{h(\lambda p, \bar{u})\} = \{h(p, \bar{u})\}$

Proof:

(a) For the proof we need to define $x^e \equiv x + [e \ 0 \ 0 \ \dots \ 0]$.

Suppose not then $px < I \quad (x \in \{x(p, I)\})$ and $\exists e$ such that $px^e \leq I$. Therefore $u(x^e) > u(x)$ which contradicts the assumption that $x \in \{x(p, I)\}$.

(b) Take $x \in \{x(p, I)\}$ and let $\bar{u} = u(x)$ for $x \in \{x(p, I)\}$.

From (a) we know that $px = I$. Now take the problem of expenditure minimization over $D \cap \{z: u(z) \geq \bar{u}\}$. x is feasible to the (E) problem since $u(x) \geq \bar{u}$ for $x \in \{x(p, I)\}$.

Now we show that no better solution for the problem (E) may exist.

Suppose that a better $x^* \in D$ could be found. This would mean

that $px^* < I$ and $u(x^*) \geq \bar{u}$. Using the same argument as in (a) this would contradict the assumed optimality of x to the (U) problem.

We have proved that:

$$\{x(p, e(p, \bar{u}))\} \subseteq \{h(p, \bar{u})\}$$

To see that the inclusion holds in reverse take $\bar{x} \in \{h(p, \bar{u})\}$. This means that $p\bar{x} = e(p, \bar{u})$. So \bar{x} is a feasible solution to the (U) problem and therefore optimal since $u(\bar{x}) \geq \bar{u}$.

We may then conclude that:

$$\{h(p, \bar{u})\} \subseteq \{x(p, e(p, \bar{u}))\}$$

and the proposition holds.

(c) The proof of (c) follows directly from the remark that $\{x(p, I)\}$ is the set of solutions to the (U) problem and that $u(x) = v(p, I)$ for any $x \in \{x(p, I)\}$ and (b).

(d), (e), (f) and (g) follow directly from the definitions of optimal solution and of optimal value function.

COROLLARY 2.1

(a) (Cournot Aggregation):

Let $x^1 \in \{x(p^1, I)\}$ and $x^2 \in \{x(p^2, I)\}$ then:

$$p^2 x^2 - p^1 x^1 = 0$$

(b) (Engel Aggregation)

Let $x^1 \in \{x(p, I^1)\}$ and $x^2 \in \{x(p, I^2)\}$ then:

$$p(x^2 - x^1) = I^2 - I^1 \quad \text{or} \quad p \Delta x = \Delta I$$

Proof: (directly from proposition 2.1.a.)

The adding up property means that the consumer expenditure equals income. This holds because expenditure cannot exceed income and if expenditure was strict less than income then (under nonsatiation and continuity of at least one good) the consumer could increase his utility by buying more of that good.

The set of Marshallian demands is equal to the set of Hicksian demands if the consumer receives a compensation in income that allows her to attain a constant level of utility (proposition 2.1.b).

Propositions 2.1.c has an analogous interpretation and follow directly from the preceding one.

Propositions 2.1.d-f (homogeneity) correspond to the intuition that there is no money illusion.

Corollary 2.1.a-b (Cournot and Engel Aggregation) are consequences of adding up.

Note that proposition 2.1. does not hold in the absence of a continuous good as we show in fig 1. and 2..

The example in figure 1 contradicts (a), (b) and (e) and the one in figure 2 contradicts (c) and (d).

We give a more familiar look to Cournot aggregation if we assume that only the first price changes.

Then we have

$$(p_1^2, \tilde{p})(x_1^2, \tilde{x}^2) - (p_1^1, \tilde{p})(x_1^1, \tilde{x}^1) = 0$$

where $\tilde{p} = (p_2, p_3, \dots, p_n)$ and p_1^j is the price of the first good in the j situation.

Rearranging we get

$$p_1^1 \Delta x_1 + \Delta p_1 x_1^1 + \Delta p_1 \Delta x_1 + \tilde{p} \Delta \tilde{x} = 0$$

if we assume differentiability in demand, divide the above equation by Δp_1 and take the limit as $\Delta p_1 \rightarrow 0$ we get

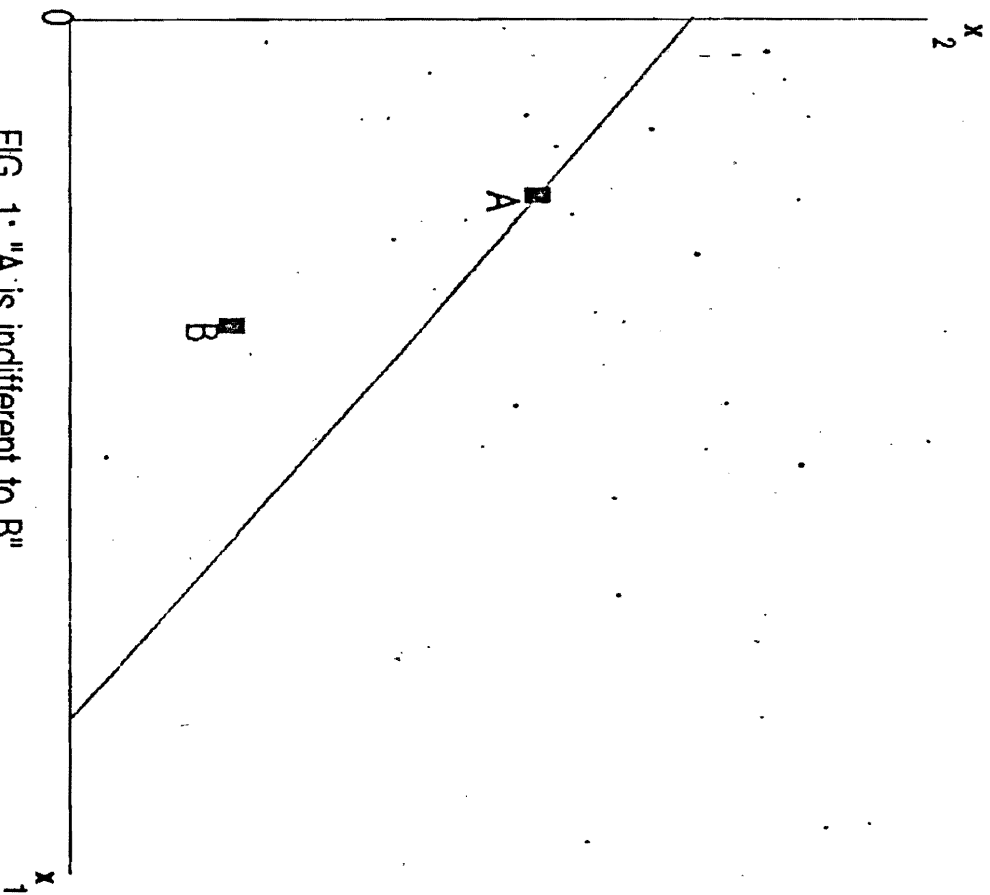


FIG. 1: "A is indifferent to B"

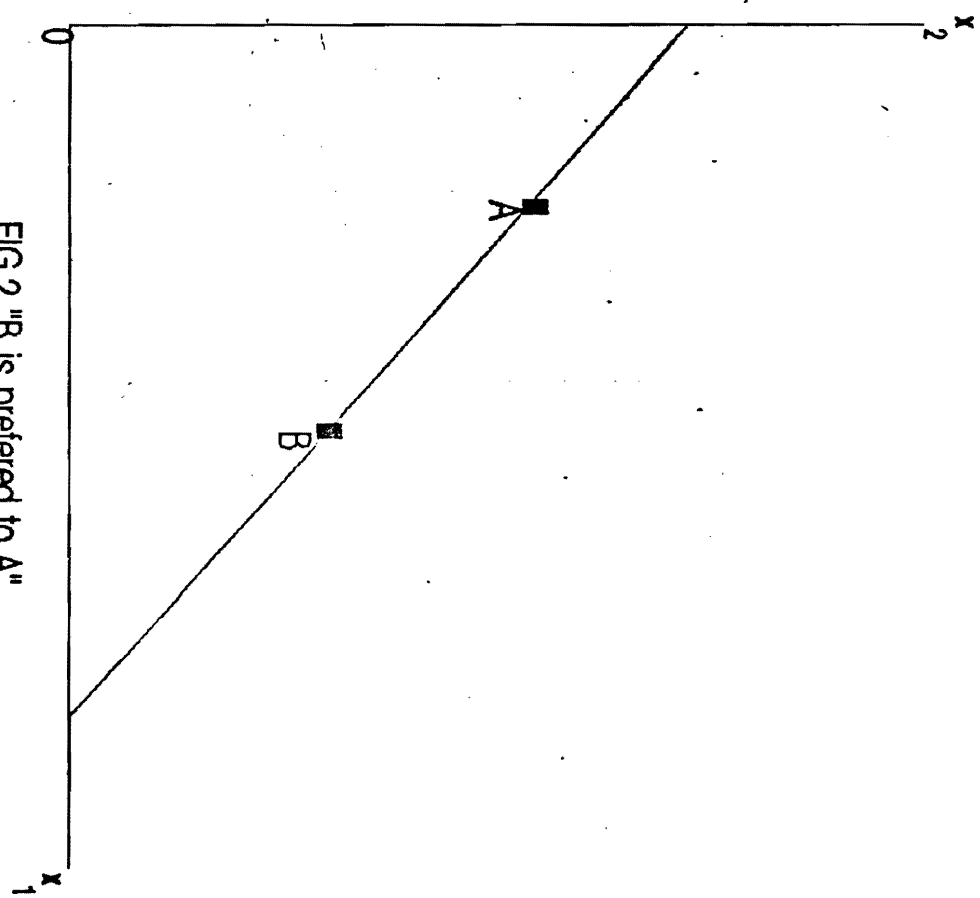


FIG. 2 "B is preferred to A"

$$x_1 + \sum_{j=1}^n p_j \frac{\partial x_j}{\partial p_1} = 0$$

which is the traditional presentation of the result.

3 - Implications of rational behavior

PROPOSITION 3.1.

$e(p, \bar{u})$ is concave (and continuous) in p .

Proof:

Define

$x^1 \in \{ h(p^1, \bar{u}) \}$, $x^2 \in \{ h(p^2, \bar{u}) \}$ and $x^\lambda \in \{ h(p^\lambda, \bar{u}) \}$
where $p^\lambda = \lambda p^1 + (1-\lambda)p^2$ and $0 \leq \lambda \leq 1$.

$$e(p^1, \bar{u}) = p^1 x^1 \leq p^1 x^\lambda$$

$$e(p^2, \bar{u}) = p^2 x^2 \leq p^2 x^\lambda \quad \text{by definition of minimum}$$

$$e(p^\lambda, \bar{u}) = p^\lambda x^\lambda = \lambda p^1 x^\lambda + (1-\lambda)p^2 x^\lambda \geq \lambda e(p^1, \bar{u}) + (1-\lambda)e(p^2, \bar{u})$$

which shows the concavity of $e(p, \bar{u})$.

The continuity follows from the concavity. (see Fleming, 1977, pp. 110-111, th. 3.5).

LEMMA 3.1.

Take p^1 and p^2 and define:

$$\xi^1 \in \partial_p e(p^1, \bar{u})$$

$$\xi^2 \in \partial_p e(p^2, \bar{u})$$

the concavity of $e(p, \bar{u})$ is equivalent to:

$$(p^1 - p^2)(\xi^1 - \xi^2) \leq 0$$

Proof: (see Clarke, 1983, p 37, pr. 2.2.9.)

PROPOSITION 3.2. (Shephard's lemma)

$$\partial_p e(p, \bar{u}) = \text{conv} \{ h(p, \bar{u}) \}$$

Proof:

(i) Let us begin by proving that $h(p, \bar{u}) \in \partial_p e(p, \bar{u})$.

$x^* \in \partial_p e(p, \bar{u})$ if, by definition:

$$x^*(p - p') + e(p, \bar{u}) \geq e(p', \bar{u}) \quad (.)$$

Take $x \in h(p, \bar{u})$ and let the vector of prices change from p to p' . As x is a feasible solution for the new problem we have by definition that (.) is verified.

(ii) Furthermore it is easy to prove that the same result holds for any $x \in \text{conv}\{h(p, \bar{u})\}$. Formally:

$$\text{conv}\{h(p, \bar{u})\} \subseteq \partial_p e(p, \bar{u})$$

(iii) We now need to prove that the reverse inclusion holds. That is:

$$\partial_p e(p, \bar{u}) \subseteq \text{conv}\{h(p, \bar{u})\}$$

We shall prove it by contradiction.

First we remark that as D is compact and $u(.)$ is continuous the set $\{x(p, e(p, \bar{u}))\}$ is a closed set.

Suppose there exists a $\tilde{x} \in \partial_p e(p, \bar{u})$ not in $\text{conv}\{x(p, e(p, \bar{u}))\}$. Then we know (see Rockafellar (1970)) that there exists an hyperplane strictly separating \tilde{x} from $\text{conv}\{x(p, e(p, \bar{u}))\}$, i.e., there is a scalar α and a nonzero vector d such that:

$$dx \geq \alpha \text{ for all } x \in \text{conv}\{x(p, e(p, \bar{u}))\}$$

and

$$d\tilde{x} < \alpha$$

Now because of PROPOSITION 3.3 (below) we know there exists a $x \in \text{conv}\{x(p, e(p, \bar{u}))\}$ such that:

$$e'(p, \bar{u}; d) \geq dx$$

thus we must have $e'(p, \bar{u}; d) \geq \alpha$.

On the other hand by PROPOSITION 3.3., we have:

$$e'(p, \bar{u}; d) = \min \{ dx : x \in \partial_p e(p, \bar{u}) \} \leq d\tilde{x} < \alpha$$

a contradiction.

COROLLARY 3.2. (Negativity)

$$(p^1 - p^2) (h(p^1, \bar{u}) - h(p^2, \bar{u})) \leq 0$$

Proof: from lemma 3.1 and proposition 3.2..

Several authors say that the concavity of the expenditure function is non intuitive. (e.g. Diewert, 1982, p. 539). Nonetheless we think that we give an intuitive presentation of the result.

To be able to present a graphical illustration we consider a change only in the price of the first good (see fig. 3.).

In the initial situation we have:

$$x^1 \in \{ h(p^1, \bar{u}) \}$$

and, by definition:

$$p^1 x^1 = e(p^1, \bar{u}) \quad (*)$$

Consider now a variation in the price of the first good (Δp_1^1). The set of solutions to the initial problem remain possible to the new problem. So we have, again by definition:

$$(p_1^1 + \Delta p_1^1, \tilde{p}) (x_1^1, \tilde{x}^1) \geq e(p_1^1 + \Delta p_1^1, \bar{u}) \quad x^1 \in \{ h(p^1, \bar{u}) \}$$

subtracting (*) from the above we get:

$$\Delta p_1^1 x_1^1 \geq \Delta e(., \bar{u}) \quad (**)$$

in particular the inequality in (**) will be strict if there are favourable substitution possibilities. From (*) and (**) one may see that the line $a + x_1^1 p_1$ (with $a = \sum_{j=2}^n p_j x_j^1$) is never below $e(., \bar{u})$ and is equal to $e(., \bar{u})$ at point p^1 (see fig. 3.).

This is the intuitive content of the above propositions.

The intuition may be stated more simply by saying that when the price of the first good increases the compensated demand for the good can not increase. If there are favourable substitution possibilities it may actually decrease.

To present the analogue to the Slutsky equation we need to

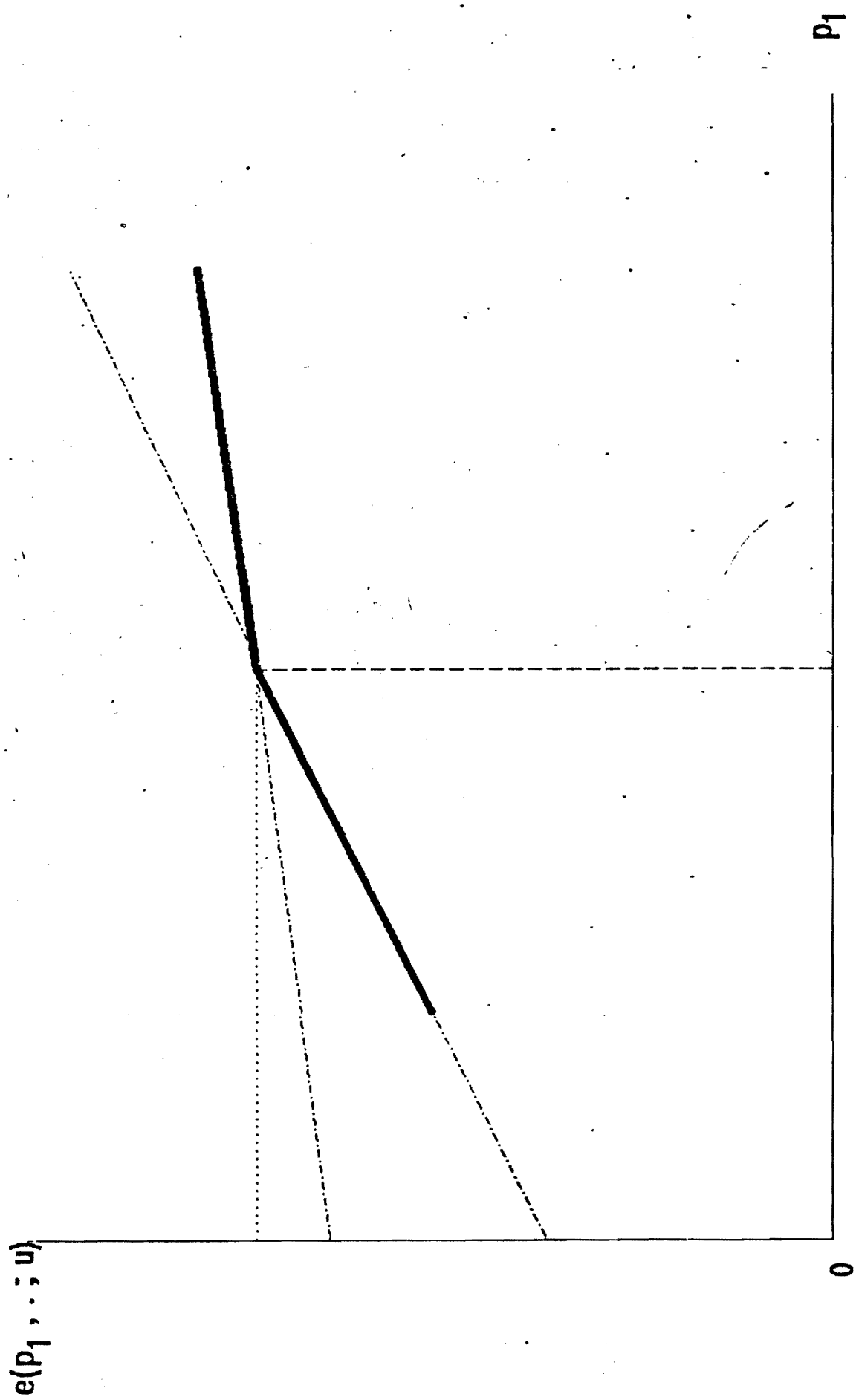


FIG.3

introduce the notion of a (one-sided) directional derivative of the expenditure function, which we shall denote as $e'(p, \bar{u}; d)$ and define as:

$$e'(p, \bar{u}; d) = \lim_{\lambda \downarrow 0} \frac{e(p + \lambda d, \bar{u}) - e(p, \bar{u})}{\lambda}$$

PROPOSITION 3.3.

$$e'(p, \bar{u}; d) = \min \{ dx : x \in \partial_p e(p, \bar{u}) \}$$

Proof:

The proof is presented in 3 steps:

(i) We shall establish that:

$$e'(p, \bar{u}; d) \geq dx \text{ for some } x \in \{x(p, e(p, \bar{u}))\}$$

Consider a sequence $\{\lambda_k\} \downarrow 0$ and price vectors of the form $p + \lambda_k d$ and let $x_k \in \{h(p + \lambda_k, \bar{u})\}$.

By definition $x_k \in \{x \in D \text{ and } u(x) \geq \bar{u}\}$ for all k and as this set is compact, we know that there is a subsequence $\{x_{k_j}\}$ $k_j \in K$ converging to a point $\tilde{x} \in \{x \in D \text{ and } u(x) \geq \bar{u}\}$.

We shall show now this limit point must be such that $\tilde{x} \in \{h(p, \bar{u})\}$.

We know that for any $k \in K$ we shall have

$$(p + \lambda_k d) x_k \leq (p + \lambda_k) x \quad \text{for any } x \in \{x \in D \text{ and } u(x) \geq \bar{u}\}$$

Now taking the limit as $k \rightarrow \infty$ we have

$p\tilde{x} \leq px$ for any $x \in \{x \in D \text{ and } u(x) \geq \bar{u}\}$ which together with the fact that $\tilde{x} \in \{x \in D \text{ and } u(x) \geq \bar{u}\}$ shows $\tilde{x} \in \{h(p, \bar{u})\}$.

By definition we have

$$e(p + \lambda_k, \bar{u}) - e(p, \bar{u}) = (p + \lambda_k) x_k - e(p, \bar{u}) \geq \lambda_k dx_k \quad \text{the inequality holding because } px_k \geq e(p, \bar{u}) \text{ (as } x_k \text{ is feasible for}$$

the (E) problem that gives $e(p, \bar{u})$).

Thus $e(p + \lambda_k \bar{u}) - e(p, \bar{u}) / \lambda_k \geq dx_k$

and taking the limit as $k \rightarrow \infty$ we have

$e'(p, \bar{u}; d) \geq d\tilde{x}$ and $\tilde{x} \in \{x(p, \bar{u})\} = \{x(p, e(p, \bar{u}))\}$ by 2.1.b.

(ii) $\partial_p e(p, \bar{u}) \supseteq \text{conv} \{x(p, e(p, \bar{u}))\}$

Let us take any $x \in \{x(p, e(p, \bar{u}))\} \equiv \{h(p, \bar{u})\}$ by 2.1.b.

so (1) $px = e(p, \bar{u})$ (by nonsatiation and definition)

And because x is feasible we have (2) $\bar{p}x \geq e(\bar{p}, \bar{u})$.

Subtracting (1) from (2) we get

$$e(\bar{p}, \bar{u}) - e(p, \bar{u}) \leq x(\bar{p} - p) \quad (3)$$

So, as $e(p, \bar{u})$ is concave, we have shown that every $x \in \{x(p, e(p, \bar{u}))\}$ is a subgradient of $e(p, \bar{u})$ at p .

It is easily shown that if every $x \in \{x(p, e(p, \bar{u}))\}$ is a subgradient then any convex combination of those points also verifies (3) and so we have

$$\text{conv} \{x(p, e(p, \bar{u}))\} \subseteq \partial_p e(p, \bar{u}) \quad \text{which ends ii)}$$

(iii) We shall now conclude the proof:

Take any $x \in \{x(p, e(p, \bar{u}))\}$. Because of ii) $x \in \partial_p e(p, \bar{u})$ then from i)

$$e'(p, \bar{u}; d) \geq \min \{dx : x \in \partial_p e(p, \bar{u})\}$$

Now take any $x \in \partial_p e(p, \bar{u})$. As $e(p, \bar{u})$ is concave we have

$$e(p + \lambda d, \bar{u}) - e(p, \bar{u}) \leq \lambda dx$$

dividing by λ and taking the limit as $\lambda \downarrow 0$ one gets

$$e'(p, \bar{u}; d) \leq dx \text{ for any } x \in \partial_p e(p, \bar{u})$$

and then

$$e'(p, \bar{u}; d) \leq \max \{dx : x \in \partial_p e(p, \bar{u})\}$$

what completes the proof.

It is simply to interpret prop. 3.3. using again fig.3.. We assume that the only price that varies is the first one. In particular suppose the first price increases. Therefore $d' = [1 \ 0 \ \dots \ 0]$. Then:

$$e'(p, \bar{u}; [1 \ 0 \ \dots \ 0]) = \min \{ x_1 : x \in \partial_p e(p, \bar{u}) \}$$

but

$$\begin{aligned} \min \{ dx : x \in \partial_p e(p, \bar{u}) \} &= \\ &= \min \{ dx : x \in \text{conv} \{ h(p, \bar{u}) \} \} = \dots \text{by prop. 3.2.} \\ &= \min \{ dx : x \in \{ h(p, \bar{u}) \} \} = \dots \text{because } dx \text{ is linear} \\ &= \min \{ dx : x \in \{ x(p, e(p, \bar{u})) \} \} \dots \text{by prop. 2.1.b.} \end{aligned}$$

So in the above case:

$$e'(p, \bar{u}; [1 \ 0 \ \dots \ 0]) = \min \{ x_1 : x \in \{ x(p, e(p, \bar{u})) \} \}$$

This result means that when the price of the first good increases (by one "small" unitary amount) the income compensation needed to assure the consumer the initial utility level is just the quantity of the first good in the demand consumption bundle that includes less of it.

The prop. 3.3. is a fundamental result that allows to construct a discrete case version of Slutsky equation.

The proof of the proposition 3.4. (Slutsky equation) is rather long and so will be done using two lemmas.

LEMMA 3.2.: Consider any sequence $\{\lambda_n\}_{n \in \mathbb{N}} \downarrow 0$. There exists a sub-sequence $\{\lambda_k\}_{k \in \mathbb{N}}$ and $k_0 \in \mathbb{N}$ such that, for every $\varepsilon > 0$, $\|x^* - x_k\| \leq \varepsilon$, $k > k_0$, $k \in \mathbb{N}$ for some $x^* \in \{x(p, e(p, \bar{u}))\}$ and some $x_k \in \{x(p + \lambda_k d, e(p + \lambda_k d, \bar{u}))\}$

Proof:

Take an arbitrary $x \in \{x(p, e(p, \bar{u}))\}$ and for every $n \in \mathbb{N}$ consider some consumption bundle $x_n \in \{x(p + \lambda_n d, e(p + \lambda_n d, \bar{u}))\}$.

As we are working in compact sets we know that there exists a convergent subsequence $\{x_k\}_{k \in K}$. Let us call x^* its limit.

Now consider the utility maximization problem (U) at prices $p + \lambda_k d$. Because of non satiation we know that the optimal consumption bundles are on the budget constraint. Therefore we have:

$$(p + \lambda_k d) x_k = e(p + \lambda_k d; \bar{u}) \text{ for any } k \in K$$

Now if we take the limit as $\lambda_k \downarrow 0$ along this subsequence we have:

$$px^* = e(p, \bar{u})$$

because the expenditure function is continuous (concave) on prices.

So we know that the limit x^* is on the budget constraint for the utility maximization problem (U) at prices p .

We proceed showing that x^* is an optimal solution to that problem. We do it by contradiction.

Take any optimal bundle at prices p , $x \in \{x(p, e(p, \bar{u}))\}$. We know we must have $px = e(p, \bar{u})$.

Because of nonsatiation in good 1 we can build a sequence of bundles \bar{x}_k , $k \in K$, of the form:

$$\bar{x}_k = x + \begin{bmatrix} \delta_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

in such a way that \bar{x}_k is feasible for the problem at prices $(p + \lambda_k d)$.

In fact, if we take

$$\delta_k = \frac{e(p + \lambda_k d, \bar{u}) - (p + \lambda_k d)x}{p_1 + \lambda_k d_1} \quad k \in K$$

we have $(p + \lambda_k d)\bar{x}_k = e(p + \lambda_k d, \bar{u})$, and, because of the continuity of the expenditure function and the fact that $px = e(p, \bar{u})$, we also have:

$$\lim_{k \in K} \bar{x}_k = x$$

Now, as x_k is optimal for the problem at prices $p + \lambda_k d$ we must have:

$$U(\bar{x}_k) \leq U(x_k)$$

and taking the limit on $k \in K$ one gets, because of the continuity of the utility function:

$$U(x) \leq U(x^*)$$

But as x^* is feasible at prices p we must have $U(x) = U(x^*)$ and $x^* \in (x(p, e(p, \bar{u})))$.

Finally we establish that the lemma holds because:

$$x^* = \lim_{k \in K} x_k$$

□

LEMMA 3.3.: Take

$$x_\lambda \in \{x(p + \lambda d, e(p, \bar{u}) + \lambda e'(p, \bar{u}; d))\} \quad \text{and}$$

$$h_\lambda \in \{x(p + \lambda d, e(p + \lambda d, \bar{u}))\}$$

and let $q_\lambda = h_\lambda - x_\lambda$.

If $\{x(p, I)\}$ is a singleton then we have:

$$\lim_{\lambda \downarrow 0} \|q_\lambda\| = 0$$

Proof:

First we prove that $\lim_{\lambda \downarrow 0} \|q_\lambda\| = 0$ exists. Because of Cournot Aggregation we know that:

$$(p + \lambda d) q_\lambda = e(p + \lambda d, \bar{u}) - e(p, \bar{u}) - \lambda e'(p, \bar{u}; d)$$

Because of the definition of directinal derivative we know that

$$\lim_{\lambda \downarrow 0} (p + \lambda d) q_\lambda = 0$$

Therefore we must have $\lim_{\lambda \downarrow 0} q_\lambda = q^*$ such that $p q^* = 0$, so

$$\lim_{\lambda \downarrow 0} \|q_\lambda\| = \|q^*\|$$

Now because of Lemma 3.2. we know that there exists a sub-sequence along wich $x_\lambda \downarrow x^* \in \{x(p, I)\}$.

Now along this same sub-sequence we have:

$$h_\lambda = x_\lambda + q_\lambda$$

and if we take the limit on the right hand side (which we know exists) we have:

$$h^* = \lim_{\lambda \downarrow 0} (x_\lambda + q_\lambda) = x^* + q^* \\ \text{(sub-seq.)}$$

We now show that $h^* \in \{x(p, I)\}$. Of course h^* is on the budget set for this problem because:

$$ph^* = px^* + pq^* = px^* = I \quad (x^* \in \{x(p, I)\})$$

On the other hand h_λ is an optimal solution to the following problem:

$$\begin{array}{ll} \min & (p + \lambda d)x \\ \text{s.t.} & \end{array}$$

$$u(x) \geq \bar{u}$$

$$x \in D$$

so we know that $u(h_\lambda) \geq \bar{u}$ and by continuity of $u(\cdot)$, we have $u(h^*) \geq \bar{u}$ which, together with the fact that $ph^* = I$ shows that $h^* \in \{x(p, I)\}$.

Now as $\{x(p, I)\}$ is a singleton, we must have $x^* = h^*$ and so $\|q^*\| = 0$ and the proof is terminated. \square

PROPOSITION 3.4. (Slutsky Equation):

$$\lim_{\lambda \downarrow 0} \|q_\lambda\| = 0$$

that is

$$\lim_{\lambda \downarrow 0} \|h_\lambda - x_\lambda\| = 0$$

for

$$x_\lambda \in \{x(p + \lambda d, e(p, \bar{u}) + \lambda e'(p, \bar{u}; d))\} \quad \text{and}$$

$$h_\lambda \in \{x(p + \lambda d, e(p + \lambda d; \bar{u}))\}$$

Proof:

If $\{x(p, I)\}$ is a singleton then Lemma 3.3. provides the proof.

Now if $\{x(p, I)\}$ is not a singleton the same happens to $\{h(p, \bar{u})\}$ and the expenditure function is not differentiable.

But as $e(p, \bar{u})$ is concave we know (see Rockafellar(1970), Th.25.5) that it is differentiable except on a set of measure zero Π .

Moreover there exists $\bar{\eta} > 0$ such that the expenditure function is differentiable at $p + \lambda d$ for $\lambda \in [\eta, \bar{\eta}] \setminus \Lambda$, for any $0 < \eta < \bar{\eta}$, where $\Lambda \subseteq \mathbb{R}_+$ is a set with at most a countable number of points (see Rockafellar (1970), th 25.3).

This fact of course implies that $\{x(p + \lambda d, I)\}$ is a singleton for $\lambda \in [\eta, \bar{\eta}] \setminus \Lambda$.

We now use lemma 3.3. to show that we must have:

$$\lim_{\lambda \downarrow \eta} \|q_\lambda\| = 0$$

for an arbitrarily small $\eta > 0$.

Suppose, by contradiction, that:

$$\lim_{\lambda \downarrow \eta} \|q_\lambda\| = L > 0$$

This would mean that, for every $\epsilon > 0$ there would exist a λ_0 such that:

$$\|q_\lambda\| \in (L - \epsilon, L + \epsilon) \text{ for } \lambda \in (0, \lambda_0)$$

Now consider $\eta_0 \in (0, \lambda_0) \setminus \Lambda$. (This point must exist because Λ has, at most, a countable number of points).

We know that for any $\lambda \in (\eta_0, \lambda_0)$ we have:

$$\|q_\lambda\| \in (L - \varepsilon, L + \varepsilon)$$

which contradicts the fact that, because of lemma 3.3., we must have:

$$\lim_{\lambda \downarrow \eta_0} \|q_\lambda\| = 0$$

The contradiction establishes the proposition. \square

We named the above proposition Slutsky equation because it permits us to construct the Hicksian demands from the knowledge of the Marshallian demands.

Remember that we need to be able to link ordinary and compensated demands to give empirical content to the negativity property above.

To give more familiar look to the expression above we assume that the only price that varies is the first one. We also assume that the solution to the consumer's problems (U) and (E) is unique. In particular suppose the first price increases. Therefore $d = [1 \ 0 \ \dots \ 0]$. Then:

$$\lim_{\lambda \downarrow 0} [h(p+\lambda d, \bar{u}) - x(p+\lambda d, e(p, \bar{u})) + \lambda \min \{x_i : x \in \{x(p, e(p, \bar{u}))\}\}] = 0$$

with $d = [1 \ 0 \ \dots \ 0]$. By PROPOSITION 3.4..

Rearranging:

$$\lim_{\lambda \downarrow 0} h(p+\lambda d, \bar{u}) = \lim_{\lambda \downarrow 0} [x(p+\lambda d, e(p, \bar{u})) + \lambda \min \{x_i : x \in \{x(p, e(p, \bar{u}))\}\}]$$

subtracting $h(p, \bar{u})$, using proposition 2.1.b., assuming differentiability of $h(.,.)$ and $x(.,.)$ and dividing by λ , we get:

$$\begin{aligned} \lim_{\lambda \downarrow 0} [h(p+\lambda d, \bar{u}) - h(p, \bar{u})/\lambda] &= \\ &= \lim_{\lambda \downarrow 0} [x(p+\lambda d, e(p, \bar{u}) + \lambda x_1) - x(p, e(p, \bar{u}))/\lambda] \end{aligned}$$

which by definition is:

$$\frac{\partial h}{\partial p_1} = \frac{\partial x}{\partial p_1} + x_1 \frac{\partial x}{\partial I}$$

which is the Slutsky equation as traditionally presented.

To present the analogue to Roy's Identity we need some preliminary results. The first one which is interesting in its own right we state as:

PROPOSITION 3.5.

$v(p, I)$ is quasi-convex in p .

Proof:

We need to prove that if :

$$v(p^1, I) \leq \bar{u} \quad \text{and} \quad v(p^2, I) \leq \bar{u} \quad \text{then} \quad v(p^\lambda, I) \leq \bar{u}$$

where $p^\lambda = \lambda p^1 + (1-\lambda)p^2$, $0 \leq \lambda \leq 1$.

We will give a proof by contradiction. Suppose not, then:

$$v(p^\lambda, I) > \bar{u}$$

let $x^\lambda \in \{ x(p^\lambda, I) \}$ then we have:

$$p^1 x^\lambda > I \quad \text{and} \quad p^2 x^\lambda > I$$

but then:

$$\lambda p^1 x^\lambda + (1-\lambda) p^2 x^\lambda > I$$

$$p^\lambda x^\lambda > I \quad \text{a contradiction.}$$

LEMMA 3.4.

Consider the level sets of the indirect utility function:

$$P(I, \bar{u}) = \{ p: v(p, I) \leq \bar{u} \}$$

These sets are convex and its supporting hyperplanes have

the form $px=I$, $x \in \text{conv} \{x(p, e(p, \bar{u}))\}$.

Proof:

$P(.,.)$ are convex sets because $v(p,.)$ is a quasi-convex function (see proposition 3.4).

To show that the second part of lemma 3.5. holds we must show that:

(a) The hyperplanes $px = I$, $x \in \text{Conv} \{x(p, e(p, \bar{u}))\}$ pass through p_0 such that $v(p_0, I) = \bar{u}$.

(b) The set $P(I, \bar{u})$ lies on the halfspace $px \geq I$.

We first show that (a) holds. Because of nonsatiation we have $p_0 x = I$ for all $x \in \{x(p_0, I)\}$. We also know (from PROPOSITION 2.1.c.) that $I = e(p_0, \bar{u})$.

So we have:

$$p_0 x = I \text{ for all } x \in \{x(p_0, e(p_0, \bar{u}))\}$$

and it is clear that the proposition also holds for every $x \in \text{Conv}\{x(p_0, e(p_0, \bar{u}))\}$.

Now we prove that (b) holds.

Take any hyperplane:

$$px = I \quad x \in \{x(p, e(p, \bar{u}))\}$$

and choose an arbitrary price vector $\bar{p} \in P(I, \bar{u})$.

We will show that $\bar{p}x \geq I$ by contradiction.

Suppose that $\bar{p}x < I$ and consider the utility maximization problem at prices \bar{p} .

Because of nonsatiation it is possible to find a bundle $\tilde{x} \in \{x \in D: \bar{p}x \leq I\}$ having a strictly better utility than x , $u(\tilde{x}) > u(x) = \bar{u}$.

This would mean that $v(\bar{p}, I) > \bar{u}$ and then $\bar{p} \notin P(I, \bar{u})$, a contradiction.

So we must have, for every $p \in P(I, \bar{u})$, $p x \geq I$ for any $x \in \{x(p, e(p, \bar{u}))\}$ and the inequality is still valid if we take any convex combination of the bundles in $\{x(p, e(p, \bar{u}))\}$.

Thus we have that, for every $p \in P(I, \bar{u})$, $p x \geq I$ for any $x \in \text{Conv} \{x(p, e(p, \bar{u}))\}$. And the proof is terminated.

PROPOSITION 3.6. (Roy's identity)

Let $x \in \{x(p, e(p, \bar{u}))\}$ then:

$p x - I = 0$ for $I = e(p, \bar{u})$

and define $\Psi(p, I) = 0 \equiv p x - I = 0$ then:

$$x(p, I) = - \frac{\frac{\partial \Psi}{\partial p}}{\frac{\partial \Psi}{\partial I}}$$

Proof:

Directly from the implicit function theorem.

Therefore we derived analogues to the consumer demand properties presented in the introduction.

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